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THREE-DIMENSIONAL THERMAL ANALYSIS OF LAMINATED COMPOSITE PLATES

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Abstract—In this paper, results of the stress analysis of multilayered plates subject to thermal and mechanical loads are presented in the context of the three-dimensional quasi-static theory of thermoelasticity. The governing equations for a plate composed of monoclinic layers and subject to any given temperature and mechanical load distributions are derived in terms of displacements. Making use of a Navier-like approach, exact three-dimensional solutions are obtained for cross-ply and antisymmetric angle-ply laminated rectangular plates subject to thermomechanical loads. Polynomial and exponential temperature distributions through the thickness are considered. Numerical results for plates with the simply supported boundary conditions are presented. It is shown that the inplane shear stresses at the corners are unbounded when plate faces are subject to uniform heating.

1. INTRODUCTION

Temperature changes often represent a significant factor, and sometimes the predominant cause of failure of composite structures subject to severe environmental loads. Debonding of layers and longitudinal cracks in the matrix are typical failure mechanisms in composite thin-walled members due to excessive stress levels caused by thermal stresses. For instance, in fiber-reinforced composites, this is due to the fact that the thermal expansion coefficients in the direction of fibers are usually much smaller (even negative) than those in the transverse direction, resulting in high normal stresses in the layers and high shear stresses at the interfaces between layers with different fiber orientations.

Various first- and higher-order theories, originally developed for the analysis of isothermal problems of laminated plates, have been extended to include thermoelastic effects [see e.g. Stavsky (1963), Nemirovskii (1972), Shaldyrvan (1980), Bapu Rao (1979), Wu and Tauchert (1980a, b), Reddy and Hsu (1980), Khdeir and Reddy (1991), Tauchert (1986, 1991) and Noor and Burton (1992a, b)]. In these displacement based theories, C^{∞}-continuous functions are used to define the displacement variation over the whole laminate thickness. As is well known, these kinds of models cannot be used to predict accurate through-the-thickness stress distributions for isothermal bending problems [see Reddy (1989, 1995), Savoia and Reddy (1992) and Savoia *et al.* (1994)]. In the case of thermal loading, these models have to be used even more carefully. However, in none of the above referenced papers has the accuracy of the proposed formulations been assessed by comparisons with exact three-dimensional solutions, as is usually done to validate new theories for isothermal bending plate problems. This is probably due to the lack of exact solutions. To the authors' knowledge, exact solutions are available only for cross-ply cylindrical panels and doubly curved shells under prescribed temperature fields [see Huang and Tauchert (1991a, b)], for steady-state thermoelasticity problems of anisotropic slabs [see Thanjitham and Choi (1991)], and for free vibration and buckling problems of angle-ply plates subject to uniform heating [see Noor and Burton (1992a, b)].

Tanigawa *et al.* (1989, 1991) pointed out the importance of performing transient thermal stress analyses when the plate is suddenly exposed to temperature changes. Using the Laplace transform method for the temperature field and adopting the classical lamination plate theory, they showed that considering only the steady-state thermal behavior often is not conservative. Nevertheless, the classical lamination theory did not allow them to evaluate the shear stress components, which play a very important role in the failure mechanisms of composite structures.

In the present study, three-dimensional solutions of rectangular multilayered plates subject to thermomechanical loads are derived within the framework of the quasi-static theory of linear thermoelasticity. Through-the-thickness temperature distributions varying according to polynomial as well as exponential laws through each individual layer are considered, so leaving the possibility of analysing both steady-state and transient thermal conditions. Three-dimensional exact solutions are derived for cross-ply and antisymmetric angle-ply laminated plates with simply supported boundary conditions. Numerical results for steady-state and transient problems of multilayered and sandwich plates subject to prescribed temperature distributions at the top and bottom faces are presented. It is shown that the displacements are C^0-functions through the thickness of the laminate, with discontinuous derivatives at the interfaces. The displacements obtained from the present exact three-dimensional solution are often very different from linear distributions through the whole laminate thickness, as is assumed by classical and first-order theories. This is the case with thermal bending of angle-ply or symmetrically laminated plates subject to uniform temperature distribution through the thickness. Hence, in those cases the classical and first-order theories fail to give accurate solutions.

For cross-ply plates, thermal strains are shown to cause high stresses even in directions different from those of the fibers. For thermal bending of angle-ply plates, inplane normal and shear stress distributions do not vary significantly with plate dimensions, but they are found to change significantly for different lamination schemes. Maximum transverse shear stresses are found systematically at the layer interfaces. The transient behavior of a plate suddenly exposed to a temperature change is also studied, making use of the Biot variational principle [see Biot (1970)] to evaluate the temperature field in the plate.

Finally, in the last section the convergence of Navier-like solutions for plates subject to thermal loads is discussed. It is shown that, for the type of simply supported boundary conditions suggested by the approach, the inplane shear stresses at the corners of the plate turn out to be unbounded, giving rise to an unbounded inplane twisting moment.

2. BASIC RELATIONS

A laminated plate of constant thickness and constituted by L homogeneous anisotropic layers is considered. The plate is referred to a orthogonal coordinate system $\{0; x_\alpha, x_3\}$, where the x_α ($\alpha = 1, 2$) axes lie in the reference plane Ω of the plate and x_3 is in the transverse direction. The top and bottom faces of the plate and the layer interfaces, located at $x_3 = x_{3t}, x_{3b}$ and x_{3i} ($i = 1, L-1$), are denoted by Ω_t, Ω_b and Ω_i, respectively. The displacement component in the x_i ($i = 1, 3$) direction is denoted by u_i. It is assumed that the layers have a plane of thermoelastic symmetry parallel to the plate midplane. As a consequence, the coefficients of the elasticity tensor C_{ijkl}, the thermal conductivity tensor k_{ij}, and the thermal expansion tensor α_{ij} take a special form with respect to the material coordinates.

The quasi-static theory of linear thermoelasticity is adopted in the present study, i.e. the coupling between the heat conduction problem and the elasticity problem is neglected [see Boley and Weiner (1960) and Nowinski (1978)]. This amounts to neglecting the temperature changes caused by deformations due to external loads, and to consider slow time rate of change in the temperature of the body. Hence, the general thermal stress problem separates into two distinct problems to be solved consecutively; the heat conduction problem and the elasticity problem for a plate under a prescribed temperature field.

First of all, the plate is considered sufficiently extended so that its edges have no influence on the heat conduction problem. In the absence of internal heat sources, the heat conduction is governed by the following boundary-value problem for any time $t > 0$:

$$cT_{,t}+k_{\alpha\beta}T_{,\alpha\beta}+k_{33}T_{,33}=0 \tag{1a}$$

$$k_{33}T_{,3}=\bar{f}_3 \quad \text{or} \quad T=\bar{T} \quad \text{at} \quad \Omega_b,\Omega_t \tag{1b}$$

$$[k_{33}T_{,3}]=0, \quad [T]=0 \quad \text{at} \quad \Omega_i \tag{1c}$$

where $\bar{T}$ and $\bar{f}_3$ are prescribed temperatures and heat fluxes, c is the specific heat, and the bracket $[\cdot]$ denotes the jump in the enclosed argument. Moreover, unless stated otherwise, throughout the paper greek and italic subscripts assume the values (1,2) and (1,2,3), respectively, and the summation convention over repeated indices is used.

As for the elasticity problem, the strain–displacement relations and the equilibrium equations for each individual layer and at the layer interfaces are [see Savoia and Reddy (1992)]

$$2\varepsilon_{\gamma\delta}=u_{\gamma,\delta}+u_{\delta,\gamma}, \quad 2\varepsilon_{\gamma 3}=u_{\gamma,3}+u_{3,\gamma}, \quad \varepsilon_{33}=u_{3,3} \tag{2}$$

$$\sigma_{\alpha\beta,\beta}+\sigma_{\alpha 3,3}=b_\alpha, \quad \sigma_{\alpha 3,\alpha}+\sigma_{33,3}=b_3 \tag{3a}$$

$$\sigma_{\alpha 3}=q_\alpha, \quad \sigma_{33}=q_3 \quad \text{at} \quad \Omega_e \tag{3b}$$

$$[\sigma_{\alpha 3}]=0, \quad [\sigma_{33}]=0 \quad \text{at} \quad \Omega_i \tag{3c}$$

$$[u_\alpha]=0, \quad [u_3]=0 \quad \text{at} \quad \Omega_i \tag{4}$$

where the surface loads q_i are equal to q_i^t at $\Omega_e=\Omega_t$ and $-q_i^b$ at $\Omega_e=\Omega_b$, and b_i are the body forces. Finally, for layers made of monoclinic material, the Duhamel–Neumann constitutive equations of thermoelasticity can be written as [see Nowinski (1978)]

$$\begin{aligned}
\sigma_{\alpha\beta} &= C_{\alpha\beta\gamma\delta}(\varepsilon_{\gamma\delta}-\varepsilon^0_{\gamma\delta})+C_{\alpha\beta 33}(\varepsilon_{33}-\varepsilon^0_{33})\\
\sigma_{33} &= C_{33\gamma\delta}(\varepsilon_{\gamma\delta}-\varepsilon^0_{\gamma\delta})+C_{3333}(\varepsilon_{33}-\varepsilon^0_{33})\\
\sigma_{\alpha 3} &= 2C_{\alpha 3\gamma 3}\varepsilon_{\gamma 3}
\end{aligned} \tag{5}$$

where

$$\varepsilon^0_{\gamma\delta}=\alpha_{\gamma\delta}T, \quad \varepsilon^0_{33}=\alpha_{33}T \tag{6}$$

are the thermal strains in the plane of the plate and in the transverse direction x_3, respectively, and T is the temperature field measured from the reference temperature (at time $t=0$) when the plate is assumed to be stress free. Making use of eqns (2), (5) and (6), the equilibrium equations (3) can be expressed in terms of the displacement components and of the assigned temperature field as follows:

$$\begin{aligned}
C_{\alpha\beta\gamma\delta}u_{\gamma,\delta\beta}+C_{\alpha\beta 33}u_{3,3\beta}+C_{\alpha 3\gamma 3}(u_{\gamma,33}+u_{3,\gamma 3}) &= b_\alpha+(C_{\alpha\beta\gamma\delta}\alpha_{\gamma\delta}+C_{\alpha\beta 33}\alpha_{33})T_{,\beta}\\
C_{33\gamma\delta}u_{\gamma,\delta 3}+C_{3333}u_{3,33}+C_{\alpha 3\gamma 3}(u_{\gamma,3\alpha}+u_{3,\gamma\alpha}) &= b_3+(C_{33\gamma\delta}\alpha_{\gamma\delta}+C_{3333}\alpha_{33})T_{,3}
\end{aligned} \tag{7a}$$

$$C_{\alpha 3\gamma 3}(u_{\gamma,3}+u_{3,\gamma})=q_\alpha, \quad C_{33\gamma\delta}u_{\gamma,\delta}+C_{3333}u_{3,3}=q_3+(C_{33\gamma\delta}\alpha_{\gamma\delta}+C_{3333}\alpha_{33})T \quad \text{at} \quad \Omega_e \tag{7b}$$

$$[C_{\alpha 3\gamma 3}(u_{\gamma,3}+u_{3,\gamma})]=0, \quad [C_{33\gamma\delta}u_{\gamma,\delta}+C_{3333}u_{3,3}]=[C_{33\gamma\delta}\alpha_{\gamma\delta}+C_{3333}\alpha_{33}]T \quad \text{at} \quad \Omega_i. \tag{7c}$$

Note that thermal strain effects appear in the form of equivalent body forces and transverse surface loads applied at the external faces and at the interfaces of the plate. The last terms are particularly significant for laminae with highly dissimilar inplane elastic coefficients $C_{\alpha\beta\gamma\delta}$ and thermal expansion coefficients $\alpha_{\gamma\delta}$, as is the case of fiber-reinforced composites.

3. THERMALLY INDUCED DEFORMATIONS

In the framework of the quasi-static theory of thermoelasticity, any transient or steady-state thermal stress analysis can be carried out by solving the heat conduction problem first [so determining the three-dimensional temperature field $T(x_\alpha, x_3)$], and then solving the elasticity problem and determining thermal deformations and stresses from eqns (2), (5) and (6).

Making use of Fourier series expansions or Laplace transforms [see e.g. Boley and Weiner (1960), Carslaw and Jaeger (1965) and Nowinski (1978)] the heat conduction problem can be solved exactly for rectangular laminated plates [see Padovan (1975a, b) and Tanigawa *et al.* (1991)], for any given temperature distribution or thermal conditions prescribed at the top and bottom of the plate. In this paper, greater attention will be paid to the solution of the elasticity problem; a general temperature field will be considered, for which a representation by means of double Fourier series with respect to the plate coordinates x_α is used:

$$T(x_\alpha, x_3) = \sum_{m_\alpha, m_\beta = 1}^{\infty} T_{m_\alpha m_\beta}(x_3) \sin(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta) \tag{8}$$

where $\delta_\alpha = m_\alpha \pi / L_\alpha$, L_α is the length of the plate edge in the x_α direction. In eqn (8) and in the following, it is assumed that $\beta \neq \alpha$, i.e. when the free index α is equal to 1, $\beta = 2$ and vice versa. The notation adopted here is very useful to write the equations in compact form. With reference to eqn (8), the cases of polynomial or exponential temperature $T_{m_\alpha m_\beta}(x_3)$ distribution through each layer are considered:

$$T^{\mathrm{P}}_{m_\alpha m_\beta}(x_3) = \sum_{r=0}^{R} A_r (x_3)^r \tag{9a}$$

or

$$T^{\mathrm{E}}_{m_\alpha m_\beta}(x_3) = \sum_{r=1}^{R} A_r \mathrm{e}^{\tau_r x_3} \tag{9b}$$

where R is the number of terms used in the temperature representation. In Sections 4 and 5 it will be shown that these two cases arise very often when exact or approximate solutions of both transient and steady-state heat conduction problems are obtained.

3.1. *Cross-ply simply supported rectangular plates*

Consider a rectangular laminated plate consisting of any number of orthotropic layers, with principal directions of orthotropy parallel to the plate edges (cross-ply lamination scheme). Since the planes x_α–x_3 are planes of elastic symmetry for each layer, the thermoelastic coefficients with an odd number of equal indices are zero, i.e. $C_{\alpha\alpha\alpha\beta}$, $C_{\alpha 3\beta 3}$, $C_{\alpha\beta 33}$ and $\alpha_{\alpha\beta}$, where, according to the notation introduced before, $\beta \neq \alpha$.

For a rectangular plate with simply supported boundary conditions, the three-dimensional elasticity solution can be obtained by making use of a Navier-like approach, by expanding the temperature field [see eqn (8)] and the mechanical loads in terms of double trigonometric series:

$$q_\alpha(x_\alpha) = \sum_{m_\alpha, m_\beta = 1}^{\infty} Q^{\alpha t(b)}_{m_\alpha m_\beta} \cos(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta) \tag{10a}$$

$$q_3(x_\alpha) = \sum_{m_\alpha, m_\beta = 1}^{\infty} Q^{3t(b)}_{m_\alpha m_\beta} \sin(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta) \tag{10b}$$

$$b_\alpha(x_\alpha, x_3) = \sum_{m_\alpha, m_\beta = 1}^{\infty} B^\alpha_{m_\alpha m_\beta} \cos(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta) \tag{10c}$$

$$b_3(x_\alpha, x_3) = \sum_{m_\alpha, m_\beta = 1}^{\infty} B^3_{m_\alpha m_\beta} \sin(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta). \tag{10d}$$

In eqns (10a) and (10b), t and b denote the top and bottom of the plate. For each term of order m_α, m_β the displacement components are written in the form

$$\begin{aligned} u_\alpha &= U_\alpha(x_3) \cos(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta) \\ u_3 &= U_3(x_3) \sin(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta) \end{aligned} \tag{11}$$

where $U_\alpha(x_3)$ and $U_3(x_3)$ are unknown functions defined through the thickness. Making use of eqns (2), (5), (6) and (8) it is easy to verify that eqns (11) satisfy the SS-1 simply supported boundary conditions [see e.g. Whitney and Leissa (1969) and Reddy (1995)], $u_\beta = u_3 = 0, \sigma_{\alpha\alpha} = 0$ at the plate edges located at $x_\alpha = 0, L_\alpha$. Hence, substituting eqns (8) and (11) in eqns (4) and (7), the following system of non-homogeneous ordinary differential equations for each layer is obtained:†

$$\begin{aligned} -(C_{\alpha\alpha\alpha\alpha}\delta^2_\alpha + C_{\alpha\beta\alpha\beta}\delta^2_\beta) U_\alpha + C_{\alpha 3\alpha 3} U_{\alpha,33} - (C_{\alpha\alpha\beta\beta} + C_{\alpha\beta\alpha\beta})\delta_\alpha\delta_\beta U_\beta \\ + (C_{\alpha\alpha 33} + C_{\alpha 3\alpha 3})\delta_\alpha U_{3,3} = B^\alpha(x_3) + (C_{\alpha\alpha\alpha\alpha}\alpha_{\alpha\alpha} + C_{\alpha\alpha\beta\beta}\alpha_{\beta\beta})\delta_\alpha T(x_3) \\ -(C_{\gamma\gamma 33} + C_{\gamma 3\gamma 3})\delta_\gamma U_{\gamma,3} - C_{\gamma 3\gamma 3}\delta^2_\gamma U_3 + C_{3333} U_{3,33} = B^3(x_3) + (C_{\gamma\gamma 33}\alpha_{\gamma\gamma} + C_{3333}\alpha_{33}) \frac{\mathrm{d}T(x_3)}{\mathrm{d}x_3} \end{aligned} \tag{12a}$$

which are subject to the following conditions at the two external faces and at the interfaces of the plate:

$$\begin{aligned} C_{\alpha 3\alpha 3}(U_{\alpha,3} + \delta_\alpha U_3) &= Q^{\alpha t(b)} \\ -C_{\gamma\gamma 33}\delta_\gamma U_\gamma + C_{3333} U_{3,3} &= Q^{3t(b)} + (C_{\gamma\gamma 33}\alpha_{\gamma\gamma} + C_{3333}\alpha_{33})T \quad \text{at } x_3 = x_{3t(b)} \end{aligned} \tag{12b}$$

$$\begin{aligned} [U_\alpha] = [U_3] = 0, \quad [C_{\alpha 3\alpha 3}(U_{\alpha,3} + \delta_\alpha U_3)] = 0_\alpha \\ [-C_{\gamma\gamma 33}\delta_\gamma U_\gamma + C_{3333} U_{3,3}] = [C_{\gamma\gamma 33}\alpha_{\gamma\gamma} + C_{3333}\alpha_{33}]T \quad \text{at } x_3 = x_{3i}. \end{aligned} \tag{12c}$$

The solution to the differential equations (12) can be obtained by selecting a particular solution $U^{\mathrm{p}}_i(x_3)$ associated with a given temperature field and body force distributions, obtaining the complementary solution by solving the homogeneous system associated with eqns (12a) and, finally, by imposing the boundary conditions (12b,c) on the sum of the two solutions. The complementary solution can be obtained as a linear combination of the six (complex) eigensolutions having the following form:‡

$$U_\alpha(x_3) = c_\alpha e^{\lambda x_3}, \quad U_3(x_3) = c_3 e^{\lambda x_3} \tag{13}$$

where λ and (c_α, c_3) are the eigenvalues and corresponding eigenvectors of the following quadratic eigenvalue problem:

† When a repeated index appears at both sides of the equation, it is not summed; 0_α means that α is a free index in that equation. For the sake of simplicity in eqns (12)–(19) the subscripts $m_\alpha m_\beta$ in the temperature and load functions $T_{m_\alpha m_\beta}(x_3)$, $B^i_{m_\alpha m_\beta}(x_3)$ and $Q^i_{m_\alpha m_\beta}(x_3)$ are omitted.

‡ In some cases, for instance for square plates with transversely isotropic layers, the eigenvalues of eqn (14) are repeated. Consequently, the complementary solution in the form (13) is no longer valid, and must be rewritten as

$$U_\alpha(x_3) = (c^1_\alpha + c^2_\alpha x_3)\, e^{\lambda x_3}, \quad U_3(x_3) = (c^1_3 + c^2_3 x_3)\, e^{\lambda x_3}.$$

$$(C_{\alpha\alpha\alpha\alpha}\delta_\alpha^2 + C_{\alpha\beta\alpha\beta}\delta_\beta^2 - C_{\alpha 3\alpha 3}\lambda^2)c_\alpha + (C_{\alpha\alpha\beta\beta} + C_{\alpha\beta\alpha\beta})\delta_\alpha\delta_\beta c_\beta - (C_{\alpha\alpha 33} + C_{\alpha 3\alpha 3})\delta_\alpha\lambda c_3 = 0_\alpha$$

$$-(C_{\gamma\gamma 33} + C_{\gamma 3\gamma 3})\delta_\gamma\lambda c_\gamma + (C_{3333}\lambda^2 - C_{\gamma 3\gamma 3}\delta_\gamma^2)c_3 = 0. \tag{14}$$

As for the particular solution (p), the method of undetermined coefficients can be successfully employed [see Hildebrand (1976)]. For instance, for a temperature distribution varying according to a polynomial law (P) of order R through each layer (9a), the solution can be obtained, for each layer, by setting

$$U_\alpha^{\mathrm{Pp}}(x_3) = \sum_{r=0}^{R} A_{\alpha r}(x_3)^r, \quad U_3^{\mathrm{Pp}}(x_3) = \sum_{r=0}^{R-1} A_{3r}(x_3)^r. \tag{15a}$$

Substituting eqn (15a) in eqn (12a) and collecting terms of equal powers of x_3, an algebraic system of $3R-1$ equations for the unknown constants $A_{\alpha r}$ and A_{3r} is obtained. Analogously, for a temperature distribution varying according to the exponential law (E) in eqn (9b), the particular solution can be set as follows:

$$U_\alpha^{\mathrm{Ep}}(x_3) = \sum_{r=0}^{R} A_{\alpha r}\,e^{\tau_r x_3}, \quad U_3^{\mathrm{Ep}}(x_3) = \sum_{r=0}^{R} A_{3_r}\,e^{\tau_r x_3}. \tag{15b}$$

Finally, it is to be remembered that the number of algebraic equations to be solved simultaneously in order to impose the boundary conditions (12b,c) for an L-layered plate is equal to $6L$. The computational effort can be significantly reduced without introducing any approximation by utilizing the "stiffness method" of Thanjitham and Choi (1991) and Choi and Thanjitham (1991). This method suggests that we can assume the interfacial displacements of the plate as unknowns, and secondly, can write the stress continuity conditions in terms of these new variables. The problem is thus reduced to the solution of a system of algebraic equations, whose coefficient matrix is a banded and symmetric matrix of dimension $3(L+1)$.

3.2. *Antisymmetric angle-ply rectangular plates*

Now consider a simply supported rectangular plate with antisymmetric angle-ply lamination scheme. In this case, the distributions of the thermoelastic coefficients $C_{\alpha\alpha\alpha\alpha}$, $C_{\alpha\alpha\beta\beta}$, $C_{\alpha\beta\alpha\beta}$, $C_{\alpha 3\alpha 3}$, $\alpha_{\alpha\alpha}$, α_{33}, $k_{\alpha\alpha}$ and k_{33} (with $\beta \neq \alpha$) are even functions with respect to the plate midplane, whereas $C_{\alpha\alpha\alpha\beta}$, $C_{\alpha 3\beta 3}$, $C_{\alpha\beta 33}$, $\alpha_{\alpha\beta}$ and $k_{\alpha\beta}$ are odd functions. Suppose that the plate is subjected to a temperature field varying according to an antisymmetric distribution through the thickness. For instance, the plate is exposed to changes in the temperature equal to $\pm\Delta T(x_\alpha)$ at the top and bottom faces, so that the functions $T_{m_\alpha m_\beta}(x_3)$ in eqn (8) turn out to be odd functions. Hence, the following displacement representation is adopted for each term of order m_α, m_β:

$$\begin{aligned} u_\alpha &= U_\alpha^1(x_3)\cos(\delta_\alpha x_\alpha)\sin(\delta_\beta x_\beta) + U_\alpha^2(x_3)\sin(\delta_\alpha x_\alpha)\cos(\delta_\beta x_\beta) \\ u_3 &= U_3^1(x_3)\sin(\delta_\alpha x_\alpha)\sin(\delta_\beta x_\beta) + U_3^2(x_3)\cos(\delta_\alpha x_\alpha)\cos(\delta_\beta x_\beta) \end{aligned} \tag{16}$$

where U_i^1 and U_i^2 $(i = 1, 3)$ are unknown functions. The solution is obtained by means of a procedure similar to that described in Section 3.1. In particular, substituting eqns (8) and (16) in eqns (7a) and collecting terms varying with the same trigonometric law, the following system of differential equations is obtained for each layer:

$$-A_{1\alpha}U_\alpha^1 + A_{2\alpha}U_{\alpha,33}^1 - A_{3\alpha}U_\beta^1 + A_{4\alpha}U_{3,3}^2 - A_{5\alpha}U_\alpha^2 - A_{6\alpha}U_\beta^2 + A_{7\alpha}U_{\beta,33}^2 - A_{8\alpha}U_{3,3}^1$$
$$= (C_{\alpha\alpha\alpha\alpha}\alpha_{\alpha\alpha} + C_{\alpha\alpha\beta\beta}\alpha_{\beta\beta} + 2C_{\alpha\alpha\alpha\beta}\alpha_{\alpha\beta})\,\delta_\alpha T(x_3)$$

$$-A_{1\alpha}U_\alpha^2+A_{2\alpha}U^2_{\alpha,33}-A_{3\alpha}U^2_\beta-A_{4\alpha}U^1_{3,3}-A_{5\alpha}U^1_\alpha-A_{6\alpha}U^1_\beta+A_{7\alpha}U^1_{\beta,33}+A_{8\alpha}U^2_{3,3}$$
$$=(C_{\alpha\alpha\alpha\beta}\alpha_{\alpha\alpha}+C_{\alpha\beta\beta\beta}\alpha_{\beta\beta}+2C_{\alpha\beta\alpha\beta}\alpha_{\alpha\beta})\,\delta_\beta T(x_3)$$

$$-A_{4\gamma}U^1_{\gamma,3}-A_{8\gamma}U^2_{\gamma,3}-A_9U_3^2-A_{10}U_3^1+A_{11}U^2_{3,33}=(C_{\gamma\delta33}\alpha_{\gamma\delta}+C_{3333}\alpha_{33})\frac{\mathrm{d}T(x_3)}{\mathrm{d}x_3}$$

$$A_{4\gamma}U^2_{\gamma,3}+A_{8\gamma}U^1_{\gamma,3}-A_9U_3^1-A_{10}U_3^2+A_{11}U^1_{3,33}=0 \quad (17)$$

where the As are defined by

$$A_{1\alpha}=(C_{\alpha\alpha\alpha\alpha}\delta_\alpha^2+C_{\alpha\beta\alpha\beta}\delta_\beta^2),\quad A_{2\alpha}=C_{\alpha3\alpha3},\quad A_{3\alpha}=(C_{\alpha\alpha\beta\beta}+C_{\alpha\beta\alpha\beta})\,\delta_\alpha\delta_\beta$$
$$A_{4\alpha}=(C_{\alpha\alpha33}+C_{\alpha3\alpha3})\delta_\alpha,\quad A_{5\alpha}=2C_{\alpha\alpha\alpha\beta}\delta_\alpha\delta_\beta,\quad A_{6\alpha}=(C_{\alpha\alpha\alpha\beta}\delta_\alpha^2+C_{\alpha\beta\beta\beta}\delta_\beta^2)$$
$$A_{7\alpha}=C_{\alpha3\beta3},\quad A_{8\alpha}=(C_{\alpha\beta33}+C_{\alpha3\beta3})\,\delta_\beta,\quad A_9=C_{\gamma3\gamma3}\delta_\gamma^2$$
$$A_{10}=C_{\gamma3\delta3}\delta_\gamma\delta_\delta,\quad A_{11}=C_{3333}. \quad (18)$$

Similarly, substituting eqns (8) and (16) in eqns (7b, c), the top, bottom and interface conditions for the unknown functions $U_\alpha^1, U_\alpha^2, U_3^1$, and U_3^2 are found. In particular, the interface conditions take the form

$$[C_{\alpha3\alpha3}(U^1_{\alpha,3}+\delta_\alpha U_3^2)+C_{\alpha3\beta3}(U^2_{\beta,3}-\delta_\alpha U_3^1)]=0_\alpha$$
$$[C_{\alpha3\alpha3}(U^2_{\alpha,3}-\delta_\alpha U_3^1)+C_{\alpha3\beta3}(U^1_{\beta,3}+\delta_\alpha U_3^2)]=0_\alpha$$
$$[-C_{\gamma\gamma33}\delta_\gamma U_\gamma^1-C_{\gamma\delta33}\delta_\delta U_\gamma^2+C_{3333}U^2_{3,3}]=[C_{\gamma\gamma33}\alpha_{\gamma\gamma}+C_{\gamma\delta33}\alpha_{\gamma\delta}+C_{3333}\alpha_{33}]T$$
$$[C_{\gamma\gamma33}\delta_\gamma U_\gamma^2+C_{\gamma\delta33}\delta_\delta U_\gamma^1+C_{3333}U^1_{3,3}]=0. \quad (19)$$

Note that, due to the properties of the thermoelastic coefficients and of the temperature distribution mentioned before, eqns (17) and (19) make the functions U_α^1, U_3^1 and U_α^2, U_3^2 represent the odd and even parts of the displacement components u_α, u_3, respectively. Correspondingly, making use of eqns (16), it is easy to verify that the solution obtained satisfies the SS-2 boundary conditions [see e.g. Whitney and Leissa (1969) and Reddy (1995)], $u_\alpha=u_3=0$ on the plate midplane ($x_3=0$), at $x_\alpha=0, L_\alpha$.

Moreover, the complementary solution is given by the linear combination of 12 eigenfunctions, having a form similar to that reported in eqn (13), where $[(c_\alpha^1, c_\alpha^2, c_3^1, c_3^2), \lambda]$ is one of the eigenpairs of the eigenvalue problem:

$$-A_{1\alpha}U_\alpha^1-A_{5\alpha}U_\alpha^2-A_{3\alpha}U_\beta^1-A_{6\alpha}U_\beta^2-\lambda(A_{8\alpha}U_3^1-A_{4\alpha}U_3^2)+\lambda^2(A_{2\alpha}U_\alpha^1+A_{7\alpha}U_\beta^2)=0_\alpha$$
$$-A_{1\alpha}U_\alpha^2-A_{5\alpha}U_\alpha^1-A_{3\alpha}U_\beta^2-A_{6\alpha}U_\beta^1+\lambda(A_{8\alpha}U_3^2-A_{4\alpha}U_3^1)+\lambda^2(A_{2\alpha}U_\alpha^2+A_{7\alpha}U_\beta^1)=0_\alpha$$
$$-A_9U_3^2-A_{10}U_3^1-\lambda(A_{4\gamma}U_\gamma^1+A_{8\gamma}U_\gamma^2)+\lambda^2(A_{11}U_3^2)=0$$
$$-A_9U_3^1-A_{10}U_3^2+\lambda(A_{4\gamma}U_\gamma^2+A_{8\gamma}U_\gamma^1)+\lambda^2(A_{11}U_3^1)=0. \quad (20)$$

As for the particular solution, again the method of undetermined coefficients can be used. For instance, for temperature distributions varying according to a polynomial law through each layer, the solution is obtained by setting

$$U_\alpha^{1(2)\mathrm{Pp}}(x_3)=\sum_{r=0}^{R}A^{1(2)}_{\alpha r}(x_3)^r\quad U_3^{1(2)\mathrm{Pp}}(x_3)=\sum_{r=0}^{R-1}A^{1(2)}_{3r}(x_3)^r. \quad (21)$$

4. STEADY-STATE HEAT CONDUCTION PROBLEM

The steady-state problem ($T_{,t} = 0$) of a cross-ply laminated plate subject to temperature distributions $T^b(x_\alpha)$ and $T^t(x_\alpha)$ at the top (t) and bottom (b) faces can be solved easily. By expanding the prescribed temperatures in the form

$$T^{b(t)}(x_\alpha) = \sum_{m_\alpha, m_\beta = 1}^{\infty} T^{b(t)}_{m_\alpha m_\beta} \sin(\delta_\alpha x_\alpha) \sin(\delta_\beta x_\beta), \tag{22}$$

the solution of the steady-state heat conduction problem can be set in the form as in eqns (8) and (9b). In fact, substituting eqn (8) in eqns (1a), the following ordinary differential equation for each layer is obtained, with reference to the term of order (m_α, m_β) of the temperature expansion:

$$\frac{d^2 T_{m_\alpha m_\beta}}{dx_3^2} - \frac{k_{\gamma\gamma}\delta_\gamma^2}{k_{33}} T_{m_\alpha m_\beta} = 0. \tag{23}$$

The solution of eqn (23) is given by the linear combination of two exponential functions:

$$T_{m_\alpha m_\beta}(x_3) = C_1 e^{-\lambda x_3} + C_2 e^{\lambda x_3} \quad \text{where} \quad \lambda = \delta_\gamma \sqrt{\frac{k_{\gamma\gamma}}{k_{33}}}. \tag{24}$$

The unknown coefficients in eqn (24) are determined by imposing the boundary and interface conditions (1b,c) which, for the problem at hand, reduce to

$$T_{m_\alpha m_\beta} = T^{t(b)}_{m_\alpha m_\beta} \quad \text{at } x_3 = x_{3t(b)}$$

$$\left[k_{33} \frac{dT_{m_\alpha m_\beta}}{dx_3} \right] = 0, \quad [T_{m_\alpha m_\beta}] = 0 \quad \text{at } x_3 = x_{3i}. \tag{25}$$

The solution of the heat conduction problem can be obtained in a simple way for uniformly distributed temperatures $T^{b(t)}$ at the two faces of the plate. In fact, in this case the temperature varies in the x_3 direction only, and the second term at the left-hand side of eqn (23) vanishes; as a consequence, the temperature variation is linear through each individual layer. The resulting $2L$ unknown coefficients (where L is the number of layers) are obtained by imposing eqns (25). For the solution of the elasticity problem, the particular solution reduces to that given in eqn (15a), with $R = 1$.

5. A SIMPLE TRANSIENT HEAT CONDUCTION PROBLEM

Many approaches have been proposed for the analysis of the transient heat transfer in composite media. Eigenvalue approaches, complex series expansions, and integral transform techniques are commonly employed for this purpose [see, e.g. Tanigawa *et al.* (1989, 1991), Padovan (1975a, b, 1974), Reismann (1968) and Han (1987)], giving an exponential variation of temperature through the thickness. In this case, the particular solution can be set in the form of eqn (15b). When the thermoelastic response of the structure is of primary interest, variational approaches to obtain an approximate solution of the heat conduction can be successfully employed. Among them, the Biot variational principle should be mentioned, which automatically satisfies the energy conservation law as a constraint. In Example 3 of Section 6 the transient problem of a cross-ply laminated plate whose faces are suddenly exposed to a temperature change $\bar{T}$ (at the time $t = 0$), uniform over the plate external faces, is studied. Following Biot (1970), the heat transfer problem can be divided into two time steps separated by the time the heat penetrates through the whole plate thickness h. If the layers have the same thermal conductivity in the transverse direction, adopting a quadratic temperature representation, the following result is obtained:

$$\bar{t} < 0.0885 \quad (q_1 < 1): \quad T(\zeta) = \bar{T}\left(\frac{\zeta^2}{q_1^2} - \frac{2\zeta}{q_1} + 1\right) \quad \text{for} \quad 0 < \zeta < q_1$$

$$T(\zeta) = 0 \quad \text{for} \quad q_1 < \zeta < 1 \tag{26a}$$

$$\bar{t} > 0.0885: \quad T(\zeta) = \bar{T}[\zeta^2(1-q_2) - 2\zeta(1-q_2) + 1]$$

$$q_2 = 1 - \exp\left[-0.218\left(\frac{\bar{t}}{0.0885} - 1\right)\right] \quad \text{for} \quad 0 < \zeta < 1 \tag{26b}$$

where $\zeta = 2x_3/h$ is a non-dimensional transverse coordinate measured from either of the two external faces $\bar{t} = 4k_{33}t/ch^2$, and $q_1 = 3.36\sqrt{\bar{t}}$ in eqn (26a) is the non-dimensional heat penetration depth. In this case, the particular solution for the thermoelasticity problem can be set in the polynomial form reported in eqn (15a), with $R = 2$.

6. NUMERICAL EXAMPLES

The steady-state and transient thermoelastic response of sandwich and laminated plates will be analysed in this Section. Two different materials are considered, a unidirectional graphite–epoxy composite (material I) and a soft core (material II). The thermoelastic properties of the two materials are reported below.

Material I

$$E_L = 200\ \text{GPa}, \quad E_T = 8\ \text{GPa}, \quad G_{LT} = 5\ \text{GPa}, \quad G_{TT} = 2.2\ \text{GPa}$$

$$\nu_{LT} = 0.25, \quad \nu_{TT} = 0.35, \quad \alpha_L = -2 \times 10^{-6}/\text{K}, \quad \alpha_T = 50 \times 10^{-6}/\text{K}$$

$$k_L = 50\text{W}/(\text{K}\cdot\text{m}), \quad k_T = 0.5\text{W}/(\text{K}\cdot\text{m}).$$

Material II

$$E_I = 1\ \text{GPa}, \quad E_T = 2\ \text{GPa}, \quad G_{IT} = 0.8\ \text{GPa}, \quad G_{TT} = 3.7\ \text{GPa}$$

$$\nu_{IT} = 0.25, \quad \nu_{TT} = 0.35, \quad \alpha_I = \alpha_T = 30 \times 10^{-6}/\text{K}, \quad k_I = k_T = 50\text{W}/(\text{K}\cdot\text{m})$$

where subscripts L and T stand for the directions perpendicular and parallel to the fibers for material I, whereas I and T stand for inplane and transverse directions for material II. Moreover, displacement and stress components will be given according to the following non-dimensional forms:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\alpha_r \bar{T} E_r}, \quad \bar{u}_i = \frac{u_i}{\alpha_r \bar{T} h} \tag{27}$$

where $\alpha_r = 10^{-6}/^\circ\text{K}$ and $E_r = 1$ GPa are reference thermal expansion and elastic coefficients. Since the examples have been prepared with the aim of obtaining reference solutions for assessing laminated plate theories, only the first term in the temperature representation (9) has been considered ($m_1 = m_2 = 1$) in all the examples.

Example 1

The steady-state thermoelastic bending of a sandwich square plate $a \times a$ subject to a sinusoidal temperature rise at the two external faces [$m_1 = m_2 = 1$, $T_{11}^b = +\bar{T}$, $T_{11}^t = -\bar{T}$ in eqn (22)] is analysed first. The thicknesses of the soft core (material II) and the two stiff external layers (material I, orientation 0°) are equal to $0.6h$ and $0.2h$, respectively, where h is the thickness of the plate. Hence, the through-the-thickness temperature variation T_{11} is

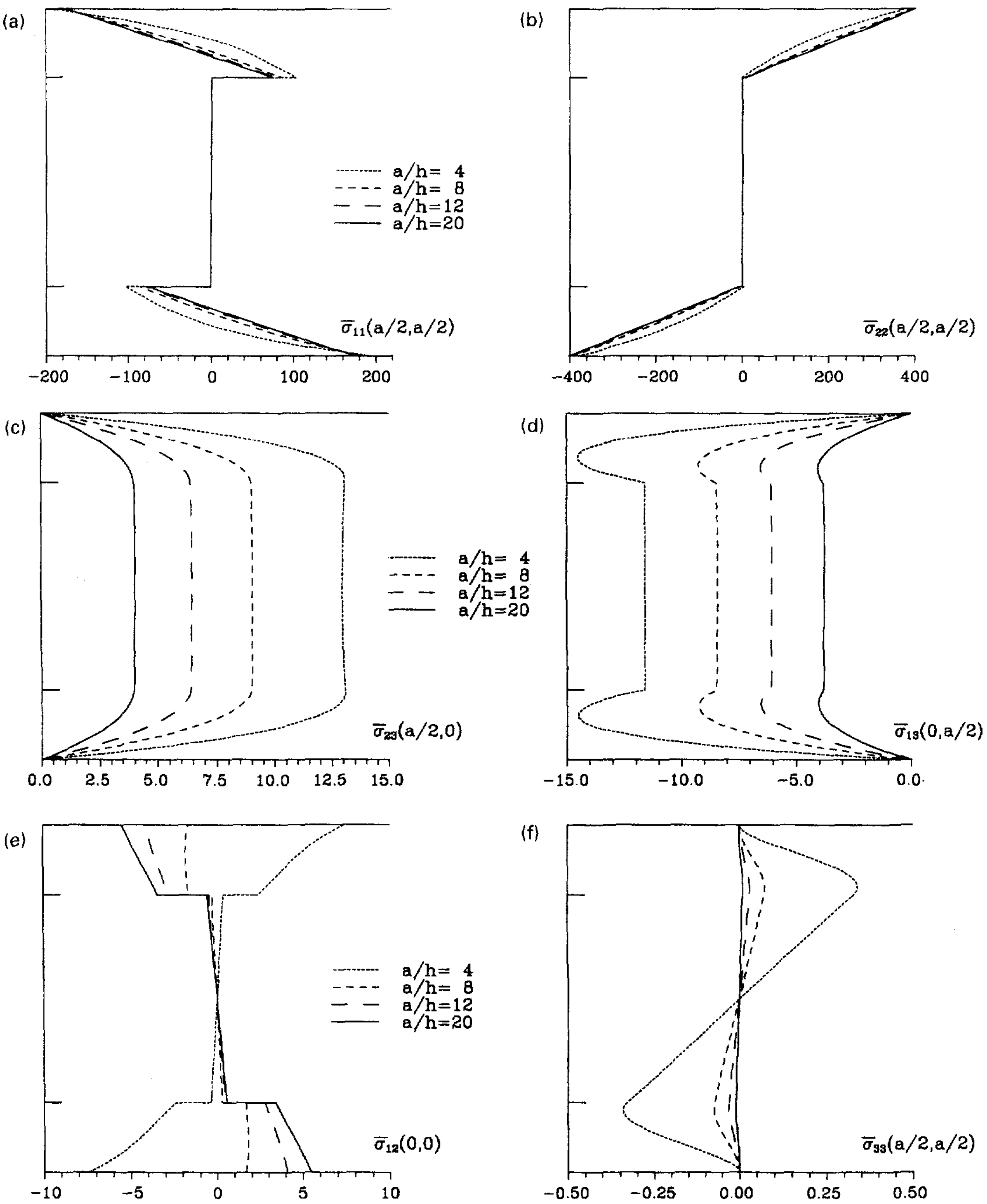

Fig. 1. Steady-state thermoelastic bending of a sandwich square plate $a \times a$ subject to a sinusoidal temperature rise at the two external faces ($m_1 = m_2 = 1, T^{t}_{11} = +\bar{T}, T^{b}_{11} = -\bar{T}$). Through-the-thickness stress distributions for different values of the aspect ratio: inplane normal stresses (a, b) $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$; transverse shear stresses (c, d) $\bar{\sigma}_{23}$ and $\bar{\sigma}_{13}$; inplane shear stress (1)(e) $\bar{\sigma}_{12}$; transverse normal stress (f) $\bar{\sigma}_{33}$.

given by the exponential law (21), which depends on the aspect ratio a/h of the plate. For thin plates the coefficient appearing on the left-hand side of eqn (23) becomes smaller and smaller, and the temperature tends to assume a linear distribution through each individual layer. Figures 1(a–f) show the plot of the normal stresses $\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$ and $\bar{\sigma}_{33}$ in the center of the plate, of the transverse shear stresses $\bar{\sigma}_{23}$ at $(a/2, 0)$ and $\bar{\sigma}_{13}$ at $(0, a/2)$, and of the inplane shear stress $\bar{\sigma}_{12}$ at the corner of the plate (0, 0) for different values of the aspect ratio. These figures show that the through-the-thickness distributions of some stress components strongly depend on the aspect ratio of the plate. For example [Fig. 1(e)], a complete reversal of the inplane shear stresses occurs when thick ($a/h = 4$) or thin ($a/h = 8$–20) plates are considered. Moreover, it is worth noting that with reference to the composite stiff faces, higher inplane stress levels occur in the direction perpendicular to the fibers [see Figs 1(a, b)] and, consequently, where the material is softer. Finally, Fig. 1(f) shows that the transverse normal stress $\bar{\sigma}_{33}$ is very small when compared with the other stress components, suggesting the possibility of developing two-dimensional refined models based on the assumption of transverse inextensibility and reduced stiffness coefficients, as presented by Savoia *et al.* (1994) for laminated plates under mechanical loads.

Example 2

This example refers to the thermal bending of a four-layered plate with antisymmetric angle-ply lamination scheme ($\theta, -\theta, \theta, -\theta$, material I for all the layers) subject to a steady-state temperature rise distributed uniformly over the plate. Since all the layers have the same thermal conductivity in the transverse direction, the temperature variation is linear through the whole plate thickness. With reference to the first term of the temperature expansion ($m_1 = m_2 = 1, T^b_{11} = +\bar{T}, T^t_{11} = -\bar{T}$), Figs 2(a–h) show the change of the stress distributions due to the change in the lamination angle θ and the aspect ratio of the plate. As for the inplane normal stresses at the center of the plate [Figs 2(a, b)] it is worth nothing that, due to the large difference between the thermal expansion coefficients in the fiber direction and in the transverse direction, $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ take different signs. As for the transverse shear stresses, $\bar{\sigma}_{13}$ at $(a/2, 0)$ presents a self-equilibrated distribution through the thickness, with maximum values at the interfaces for $\theta = 45°$. Displacement distributions are presented

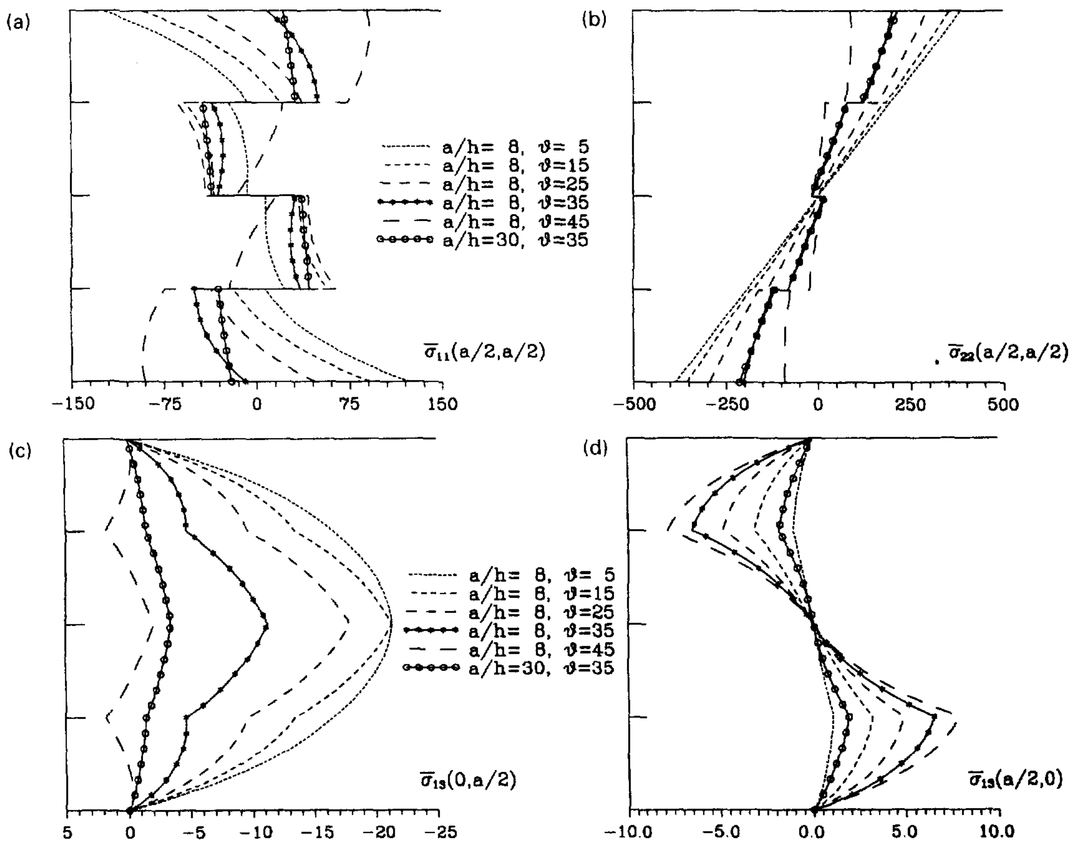

Fig. 2. Steady-state thermoelastic analysis of a four-layered angle-ply ($\theta, -\theta, \theta, -\theta$, material I for all the layers) square plate $a \times a$ subject to a temperature change at the two external faces, uniform over the plate domain. The first term in the temperature expansion is considered ($m_1 = m_2 = 1$, $T^b_{11} = +\bar{T}$, $T^t_{11} = -\bar{T}$). Non-dimensional stresses: inplane normal stresses (a, b) $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$; transverse shear stresses (c, d) $\bar{\sigma}_{23}$ and $\bar{\sigma}_{13}$; inplane shear stress (3) $\bar{\sigma}_{12}$; inplane displacements (f, g) $\bar{u}_1$ and (h) $\bar{u}_2$.

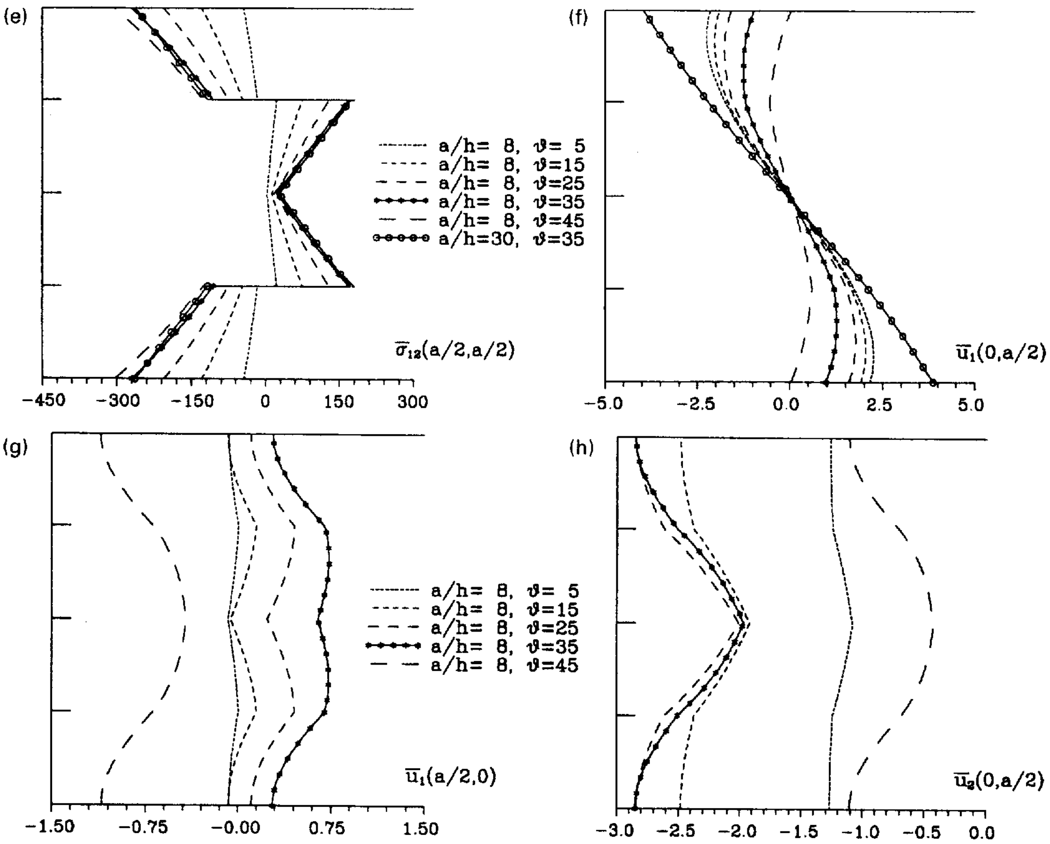

Fig. 2.—Continued.

in Figs 2(f–h). The displacements are given by C^0-functions with discontinuous derivatives at the layer interfaces, and they are different from linear distributions through the whole plate thickness. Hence, in the case the first-order theory [see e.g. Stavsky (1963), Reddy and Hsu (1980), Khdeir and Reddy (1991) and Tauchert (1986, 1991)] is expected to fail when evaluating the stress distributions.

Example 3

Finally, the transient solution of a symmetric four-layered square laminate (0°, 90°, 90°, 0°), with material I for all the layers and suddenly exposed to a uniform temperature change $\bar{T}$ (at the time $t = 0$) over the plate surfaces, is presented. The through-the-thickness distribution of the temperature [see eqn (26)], stress and displacement components are presented in Figs 3(a–h) for different times. $\bar{t} = \infty$ represents the final steady-state condition when the whole plate reaches the temperature $\bar{T}$. Even in this case, the two normal stress components take opposite signs in both the external and internal layers [Figs 3(b,c)]. Moreover, both the transverse shear stresses are given by self-equilibrated distributions through the thickness, and maximum values are found at the layer interfaces [see Figs 3(e,f)]. The displacements are quite different from the constant distributions assumed in the classical theory, not only in the transient phase, but even when the whole plate is heated. Hence, even in this case, it seems to be indispensable to use displacement representations through each individual layer in order to predict accurate stress distributions.

7. SOME REMARKS ON NAVIER-LIKE SOLUTIONS

In Section 3 a Navier-like solution for simply supported laminated plates has been derived. This approach represents the simplest case for which analytical solutions can be derived. It allows the reduction of the three-dimensional elasticity problem described by partial differential equations to one governed by a system of ordinary differential equations with respect to the transverse coordinate x_3. In the case of laminate theories, the procedure reduces a two-dimensional problem to the solution of a system of algebraic equations, provided some special simply supported boundary conditions are imposed at the edges. In fact, only in this case no boundary layers arise at the plate edges, and the problem reduces to an interior domain problem. It is worth noting that this special kind of boundary conditions can cause significant mathematical problem for plates subject to thermal loads.

For instance, consider a cross-ply, symmetrically laminated rectangular plate subject to a linear temperature distribution through the thickness. Making use of the classical laminate theory, the displacement equations of equilibrium can be reduced to

$$D_{11}w_{,1111}+2(D_{12}+2D_{66})w_{,1122}+D_{22}w_{,2222} = q_3 - M^{\mathrm{T}}_{1,11} - M^{\mathrm{T}}_{2,22} \tag{28}$$

where w is the transverse deflection, D_{ij} are the bending stiffnesses of the laminate, and M_1^{T} and M_2^{T} are the thermal moments. For an $a \times a$ square plate subject to uniformly distributed thermal moments $M_1^{\mathrm{T}}(x_1, x_2) = M^{\mathrm{T}}$ and transverse load $q_3(x_1, x_2) = q$, making use of a Navier-like approach, the following expressions are found for w, the twisting moment M_{12} and the bending moment M_{11}, respectively:

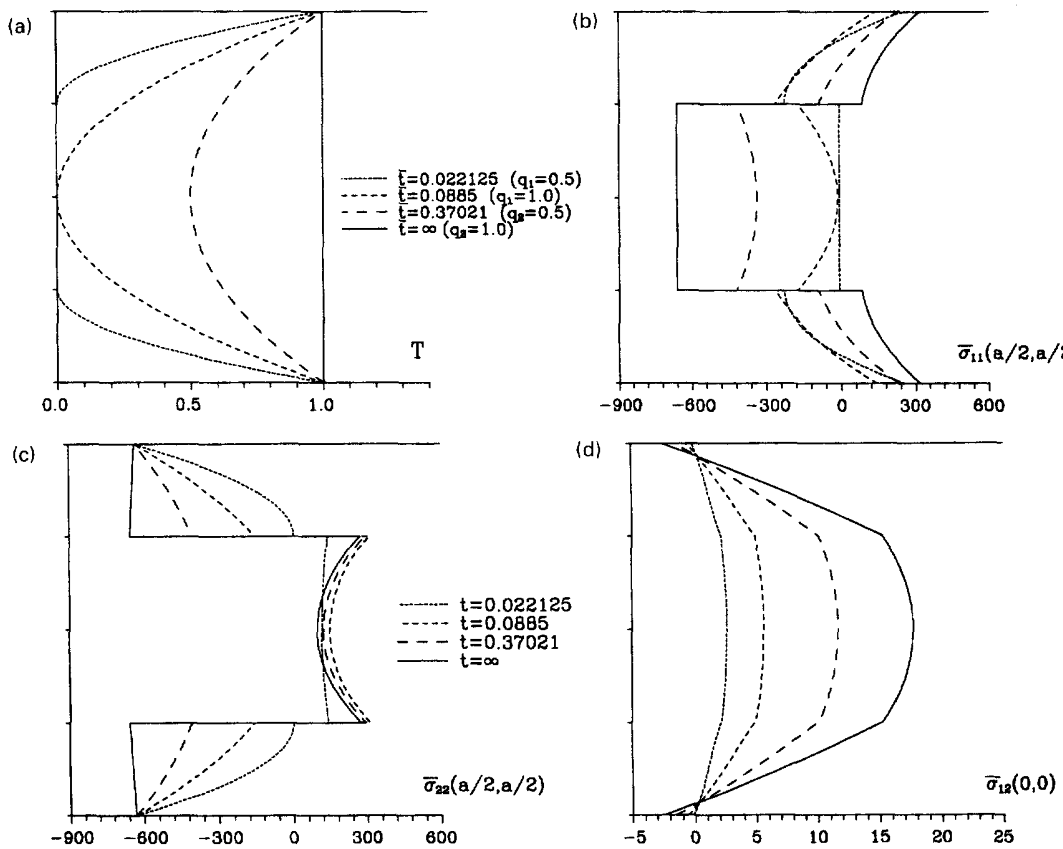

Fig. 3. Transient analysis of a cross-ply laminate (0°, 90°, 90°, 0°), with material I for all the layers ($h = 2$, $a/h = 8$), and suddenly exposed to a temperature change $\bar{T}$ (at the time $t = 0$), uniform over the plate surfaces. The first term only in the temperature expansion is considered ($m_1 = m_2 = 1$). (a) Through-the-thickness temperature distribution at different times. Non-dimensional stresses: inplane normal stresses (b, c) $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$; inplane shear stress (d) $\bar{\sigma}_{12}$; transverse shear stresses (e, f) $\bar{\sigma}_{23}$ and $\bar{\sigma}_{13}$; inplane displacements (g, h) $\bar{u}_1$ and $\bar{u}_2$.

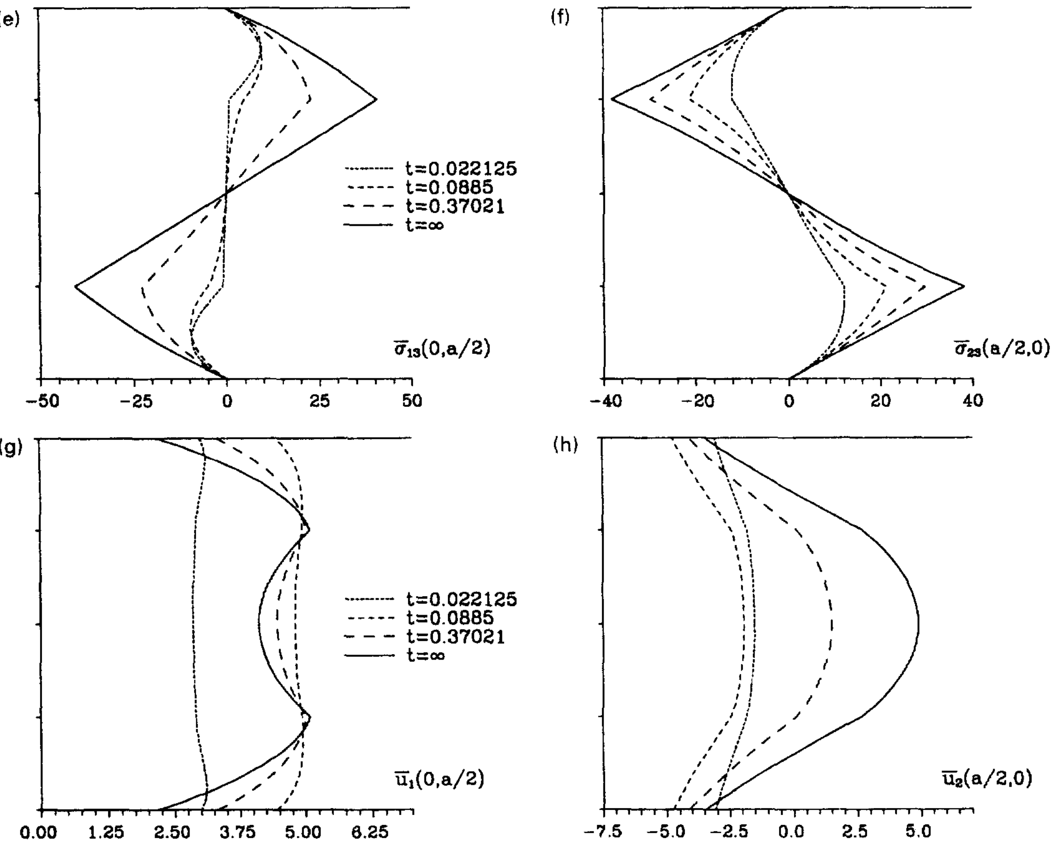

Fig. 3.—Continued.

$$w(x_1,x_2) = \frac{16a^4}{\pi^6}\sum_{n,m=1}^{\infty}\frac{1}{mn}\frac{q+\dfrac{\pi^2}{a^2}M^{\mathrm{T}}(m^2+n^2)}{D_{11}m^4+2(D_{12}+2D_{66})m^2n^2+D_{22}n^4}\sin\alpha_m x_1\sin\beta_n x_2 \quad (29a)$$

$$M_{12}(x_1,x_2) = -2D_{66}\frac{16a^4}{\pi^6}\sum_{n,m=1}^{\infty}\frac{q+\dfrac{\pi^2}{a^2}M^{\mathrm{T}}(m^2+n^2)}{D_{11}m^4+2(D_{12}+2D_{66})m^2n^2+D_{22}n^4}\cos\alpha_m x_1\cos\beta_n x_2 \quad (29b)$$

$$M_{11}(x_1,x_2) = \frac{16}{\pi^2}\sum_{n,m=1}^{\infty}\left[\frac{a^4}{\pi^4}\frac{(D_{11}m^2+D_{22}n^2)}{mn}\frac{q+\dfrac{\pi^2}{a^2}M^{\mathrm{T}}(m^2+n^2)}{D_{11}m^4+2(D_{12}+2D_{66})m^2n^2+D_{22}n^4}-\frac{M^{\mathrm{T}}}{mn}\right]\times\sin\alpha_m x_1\sin\beta_n x_2. \quad (29c)$$

It is worth noting that not all the series expressions in eqns (29) converge when thermal bending moments are present, whereas they always converge when mechanical loads only are present. For instance, consider the case $q = 0$ and $M^{\mathrm{T}} = 1$, and set $A = D_{11}, B = (D_{12}+2D_{66}), C = D_{22}$. The series appearing in the transverse displacement (29a) is absolutely convergent at each point (x_1, x_2) and, therefore, is convergent. In fact, setting $D = \min(A, B, C)$ and taking the relation $m^2+n^2 \geqslant 2mn$ into account, one obtains

$$\frac{m^2+n^2}{mn(Am^4+2Bm^2n^2+Cn^4)}\sin\alpha_m x_1 \sin\beta_n x_2 \leqslant \frac{m^2+n^2}{mn(Am^4+2Bm^2n^2+Cn^4)}$$

$$\leqslant \frac{1}{Dmn(m^2+n^2)} \leqslant \frac{1}{D}\frac{1}{m^2n^2} \tag{30}$$

which is a convergent series [see Smirnov (1964)]. It is interesting to note that, for the same loading condition, the Navier-like solution predicts unbounded twisting moment M_{12} at the corners of the plate. In fact, setting $(x_1, x_2) = (0, 0)$ and $D = \max(A, B, C)$, one obtains

$$\frac{m^2+n^2}{(Am^4+2Bm^2n^2+Cn^4)} \geqslant \frac{1}{D(m^2+n^2)} \geqslant \frac{1}{D(m+n)^2} \tag{31}$$

which diverges [see Smirnov (1964)].

By making use of the properties of the alternating series it can be shown that the twisting moment in the interior of the plate, $0 < x_1 < a$ and $0 < x_2 < b$, as well as the bending moments M_{11} and M_{22} in the whole plate domain converge to finite values when the corresponding series in eqns (29) are computed. In a similar way, it is easy to show that, when the uniform transverse load q only is considered, all of the series appearing in eqns (29) are absolutely convergent. For instance, setting $q = 1$ and $M^T = 0$ in the transverse displacement expansion presented in eqn (29a), the following is obtained:

$$\frac{1}{mn(Am^4+2Bm^2n^2+Cn^4)}\sin\alpha_m x_1 \sin\beta_n x_2 \leqslant \frac{1}{Dmn(m^2+n^2)^2} \leqslant \frac{1}{D}\frac{1}{m^3n^3} \tag{32}$$

where $D = \min(A, B, C)$. The series reported in eqn (32) converges. Note that the convergence rate of all the series presented in eqns (29) when a transverse load q is applied is faster with respect to the case when thermal bending moments M^T are assigned. For instance, the convergence of displacements and moments for an isotropic square plate $(a = b = 10, h = 1)$ for an increasing number of terms in the corresponding series is shown in Figs 4(a, b) for mechanically and thermally loaded plates, respectively. The twisting moment M_{12} has been evaluated at the corner of the plate $(0, 0)$ and at a point very close to it $(0.01a, 0.01a)$. Figure 4(b) clearly shows, for the uniformly thermally loaded plate, that the series expansion (29b) evaluated at the corner of the plate does not converge.

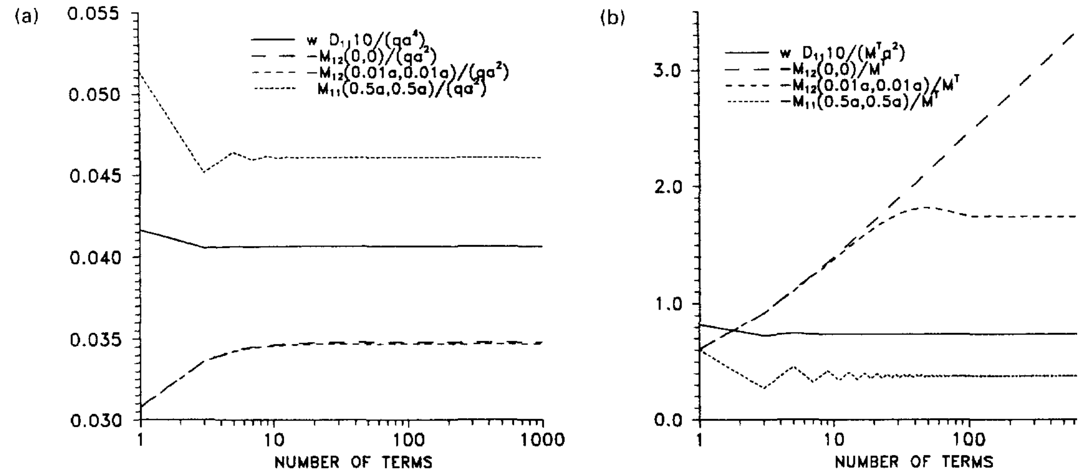

Fig. 4. Convergence of Navier-like solutions for square isotropic plates subject to mechanical (a) and thermal loads (b); transverse displacement w and bending moment M_{11} in the center of the plate; twisting moment M_{12} in the corner (0, 0) and in a point very close to it $(0.01a, 0.01a)$.

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